Travelling waves and QCD saturation

Guillaume Beuf

CEA / DSM / SPhT

Content

An introduction to the traveling wave method in gluon saturation

S. Munier, R. Peschanski (2003-04)

- Link between saturation and nonlinear physics
- Nonlinear wave fronts formation and universal properties
- Results for the Balitsky-Kovchegov equation with fixed or running coupling

A more recent result

S. Sapeta, R. Peschanski (2006), G. B., R. Peschanski (2007)

- Stability with respect to higher orders

Balitsky-Kovchegov equation

Balitsky (1996), Kovchegov (1999, 2000)

$$\partial_Y \, T_Y(\mathbf{x},\mathbf{y}) = \bar{\alpha} \int \frac{d^2\mathbf{z}}{2\pi} \, \frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2} \, \big[T_Y(\mathbf{x},\mathbf{z}) + T_Y(\mathbf{z},\mathbf{y}) - T_Y(\mathbf{x},\mathbf{y}) - T_Y(\mathbf{x},\mathbf{z})T_Y(\mathbf{z},\mathbf{y})\big]$$

- BFKL kernel $\Rightarrow$ exponential growth of $T_Y(\mathbf{x},\mathbf{y})$
- Nonlinear damping $\Rightarrow$ saturation at $T_Y(\mathbf{x},\mathbf{y}) = 1$

1D-BK equation

- Fourier transform $\mathbf{x}-\mathbf{y} \mapsto \mathbf{k}$ and $(\mathbf{x}+\mathbf{y})/2 \mapsto \mathbf{q}$
- Restriction to zero momentum transfert $q = 0$
- Rotationnal invariance $\Rightarrow \tilde{T}_Y(\mathbf{k}, \mathbf{q}=0) \equiv N(\log(k^2/Q_0^2), Y)$

$$\partial_Y \, N(L,Y) \;=\; \bar{\alpha}\left[\chi_{LL}(-\partial_L)N(L,Y) - N^2(L,Y)\right]$$

With $\chi_{LL}(\gamma) = 2\Psi(1) - \Psi(\gamma) - \Psi(1-\gamma)$.

Mapping to FKPP

Munier, Peschanski (2003)

- Diffusive approximation:
  $\chi_{LL}(-\partial_L) \simeq \chi_{LL}(\frac{1}{2}) + \frac{1}{2}\chi''_{LL}(\frac{1}{2})\left(\frac{1}{2}+\partial_L\right)^2$
- Change of variables: $t \propto \bar{\alpha} Y$, $x = C_1 L + C_2 \bar{\alpha} Y$, and $u(t,x) \propto N(L,Y)$

$\Rightarrow$ FKPP equation

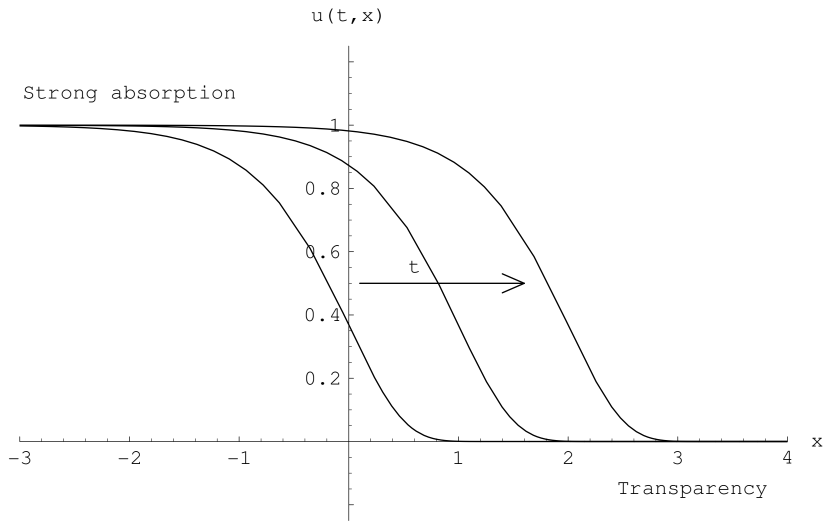
$$\partial_t u(t,x) \quad = \quad \partial_x^2 u(t,x) + u(t,x) - u^2(t,x)$$

Fisher (1937), Kolmogorov, Petrovsky, Piscounov (1937)

Uniformly translating fronts

$$u(t,x) = \phi_v(z)\ , \quad \text{with} \quad z \equiv x - vt$$

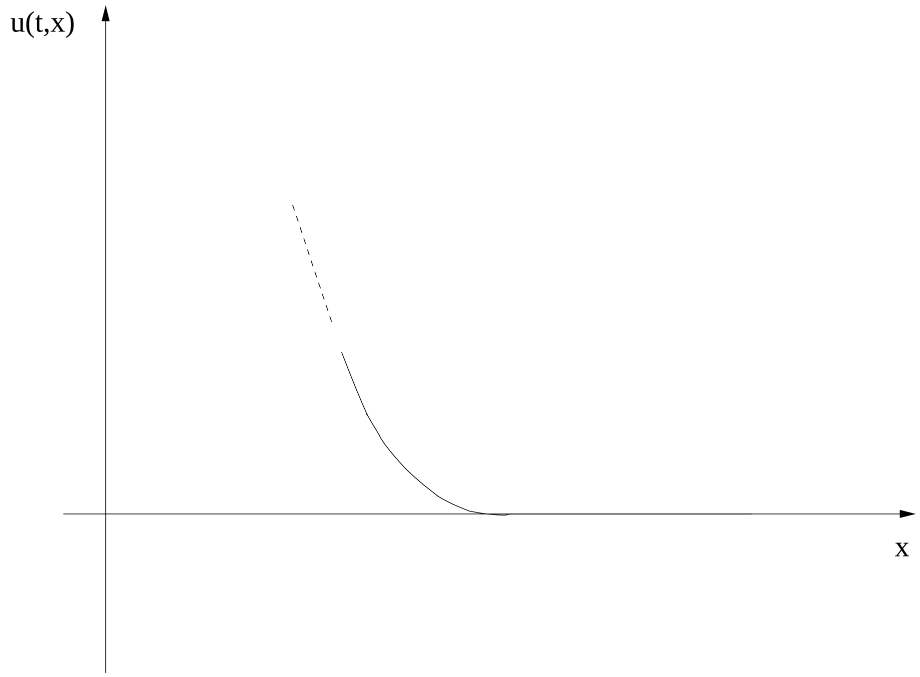
$$\begin{cases} -v\ \phi_v'(z) = \phi_v''(z) + \phi_v(z) - \phi_v^2(z) \\ \phi_v(-\infty) = 1 \quad \text{and} \quad \phi_v(+\infty) = 0 \end{cases}$$



Linear regime at large z:

$$\phi_v(z) \propto e^{-\gamma z} \qquad \text{for} \quad z \to \infty \quad \text{where} \quad v = \gamma + \frac{1}{\gamma}$$

First case: $v \geq 2$

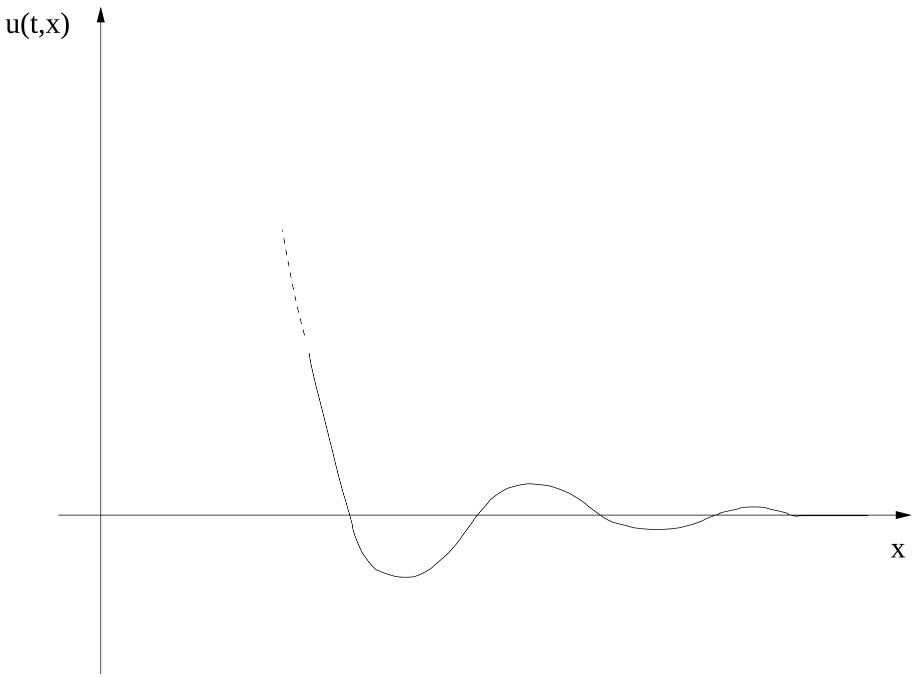


Uniformly translating fronts

Linear regime at large z:

$$\phi_v(z) \propto e^{-\gamma z} \qquad \text{for} \quad z \to \infty \quad \text{where} \quad v = \gamma + \frac{1}{\gamma}$$

Second case: $v < 2$



Uniformly translating fronts

Linear regime at large z:

$$\phi_v(z) \propto e^{-\gamma z} \qquad \text{for} \quad z \to \infty \quad \text{where} \quad v = \gamma + \frac{1}{\gamma}$$

Assumption: $u(t,x) \geq 0$

$$\Rightarrow v \geq 2$$

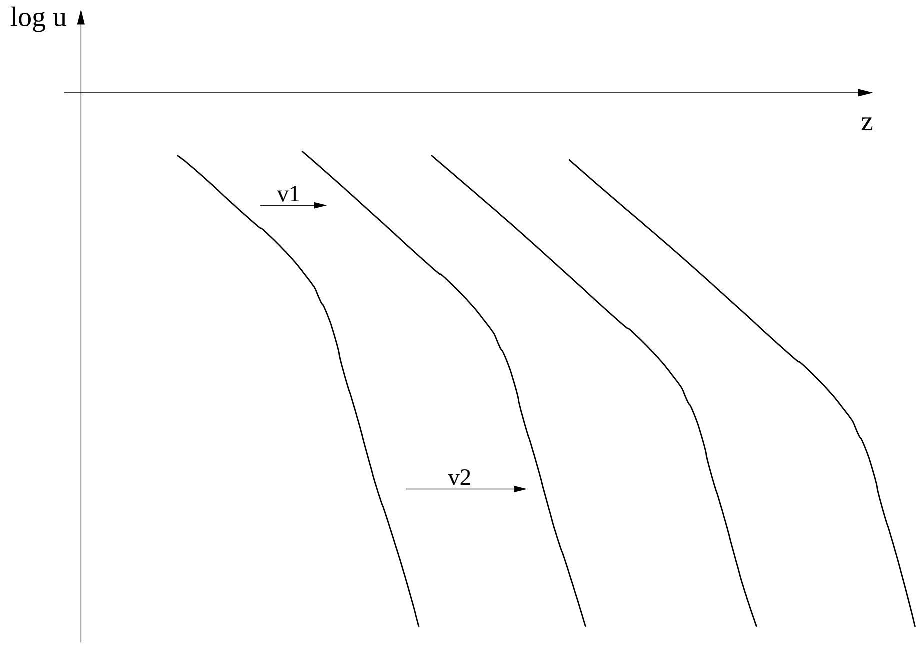
$$v = v(\gamma)\,, \quad \text{for all} \quad \gamma > 0$$

$$v_{min} = v(1) = 2$$

Generic initial condition

Local study in the linear regime at large z:

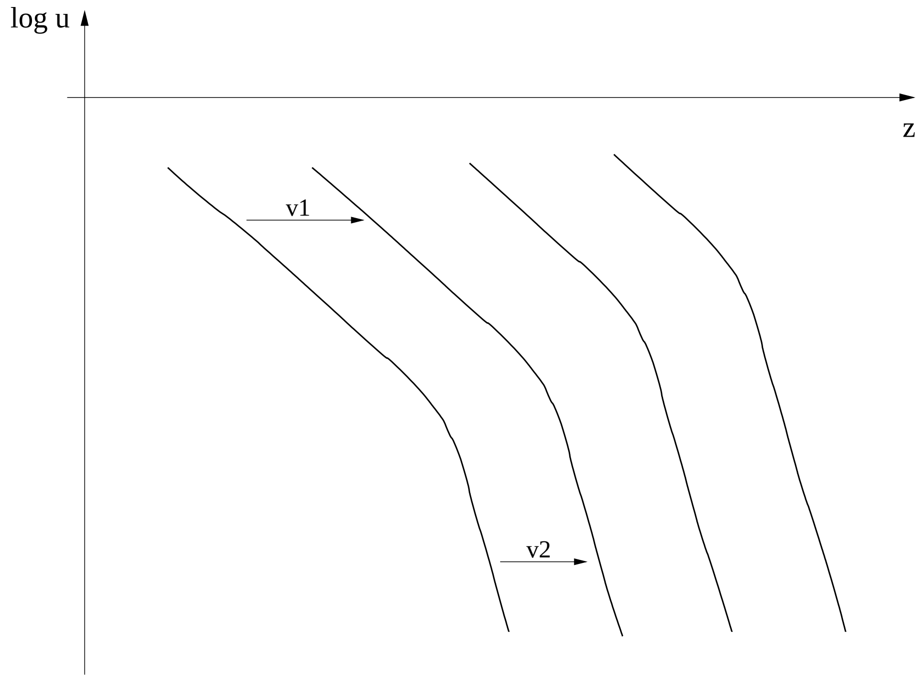
If $v_1 < v_2$



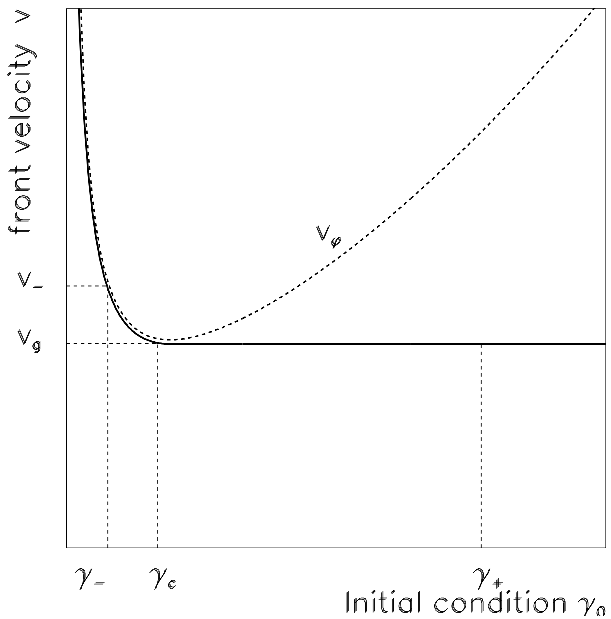
Generic initial condition

Local study in the linear regime at large z:

If $v_1 > v_2$



Asymptotic front



Nonlinear damping $\Rightarrow$ flat solution at small x.

$$\text{For } t \to \infty, \quad u(t,x) \sim \begin{cases} e^{-\gamma_0 z} & \text{if} \quad \gamma_0 < \gamma_c = 1 \\ e^{-\gamma_c z} & \text{if} \quad \gamma_0 \geq \gamma_c \end{cases} \tag{1}$$

Convergence to the asymptotic front

Ansatz:

$$\begin{aligned} u(t,x) &= t^{\alpha} G\left(\frac{\xi}{t^{\alpha}}\right) e^{-\gamma_c \xi} \\ \xi &= x - v_c t + c(t) \end{aligned}$$

for $t \to \infty$, and $\xi \leq \mathcal{O}(t^{\alpha})$.
The FKPP equation gives then

$$\begin{aligned} \alpha &= \frac{1}{2} \quad , \quad \dot{c}(t) = \frac{\beta}{t} \\ 0 &= G''(z) + \frac{z}{2}\, G'(z) + (\beta - \frac{1}{2})\, G(z) \end{aligned}$$

Boundary conditions:
- $G(z)$ bounded for $z \to \infty$.
- $G(z) \sim z$ for $z \to 0$ (nonlinearities).

Solution of the FKPP equation

For $t \to \infty$, and $\mathcal{O}(1) \leq \xi \leq \mathcal{O}(\sqrt{t})$:

$$\begin{aligned} u(t,x) &= A\,\xi\,e^{-\frac{\xi^2}{4t}}\;e^{-\xi} \\ \xi &= x - 2t + \frac{3}{2}\log t \end{aligned}$$

For $\xi \gg \sqrt{t}$: initial condition still relevant.
For $\xi \leq \mathcal{O}(1)$: nonlinear term relevant.

Universality

For an FKPP-like equation with

- an unstable homogeneous equilibrium state
- a family of uniformly translating front solutions
- an effective nonlinear damping
- a steep enough initial condition

$\Rightarrow$ Universal traveling wave asymptotic solution, independant of the precise form of

- the nonlinearities
- the initial condition

Bramson (1983), Brunet, Derrida (1997), Ebert, van Saarloos (2000)

BK critical parameters

Solution in the linear regime:

$$N(L,Y) = \int \frac{\mathrm{d}\gamma}{2\pi i}\, e^{-\gamma L + \bar{\alpha}\chi_{LL}(\gamma) Y}\, N_0(\gamma)$$

Dispersion relation:

$$v = \bar{\alpha}\, \frac{\chi_{LL}(\gamma)}{\gamma}$$

Critical parameters:

$$\begin{aligned} \chi_{LL}(\gamma_c) &= \gamma_c\, \chi'_{LL}(\gamma_c) \quad \Rightarrow \gamma_c = 0.6275\ldots \\ v_c &= \bar{\alpha}\, \frac{\chi_{LL}(\gamma_c)}{\gamma_c} \end{aligned}$$

Initial condition:

Color transparency $\Rightarrow N(L,Y) \propto e^{-L}$ for $L \to \infty$

Asymptotic solution of BK

Mueller, Triantafyllopoulos (2002), Munier, Peschanski (2003)

Universal asymptotic solution:

$$\begin{aligned} N(L,Y) &= A\,\xi\, e^{-\frac{\xi^2}{2\bar{\alpha}\chi''_{LL}(\gamma_c)Y}}\; e^{-\gamma_c\xi} \\ \xi &= L-\bar{\alpha}\,\frac{\chi_{LL}(\gamma_c)}{\gamma_c}Y+\frac{3}{2\gamma_c}\log Y\,, \end{aligned}$$

for $Y\to\infty$, and $\xi\leq\mathcal{O}(\sqrt{Y})$.

$\Rightarrow$ Geometric scaling in $\tau=e^{\xi}=\frac{k^2}{Q_0^2}\,e^{-\bar{\alpha}\,\frac{\chi_{LL}(\gamma_c)}{\gamma_c}Y}\;Y^{\frac{3}{2\gamma_c}}$,

and scaling violations if $\xi\geq\mathcal{O}(\sqrt{Y})$.

A BK equation with running coupling

Choice: let us take $\bar{\alpha}$ at the scale k_T of the parent dipole.

$$\bar{\alpha}(k_T^2) = \frac{1}{b\ \log(k_T^2/\Lambda^2)} = \frac{1}{b\ L}$$

$$bL\ \partial_Y N(L,Y) = \chi_{LL}(-\partial_L)N(L,Y) - N^2(L,Y)$$

Approximate linear solution

$$N(L,Y) = \int \frac{d\gamma}{2\pi i} \int \frac{d\omega}{2\pi i}\, e^{-\gamma L + \omega Y + \frac{1}{b\omega}\, X(\gamma)}\, N_0(\gamma,\omega)$$

with $$X(\gamma) = \int_{\hat{\gamma}}^{\gamma} d\gamma'\, \chi_{LL}(\gamma')$$

is an approximate solution of the linear equation at large L, because the large L saddle point equation is

$$L = \frac{1}{b\omega}\, \chi_{LL}(\gamma)$$

Dispersion relation

$$N(L,Y) = \int \frac{d\gamma}{2\pi i} \int \frac{d\omega}{2\pi i}\, e^{-\gamma L + \omega Y + \frac{1}{b\omega}\, X(\gamma)}\, N_0(\gamma,\omega)$$

At large Y, the saddle point approximation in ω gives

$$\begin{aligned} \omega_s &= \sqrt{\frac{X(\gamma)}{bY}} \\ N(L,Y) &\sim \int \frac{d\gamma}{2\pi i}\, e^{-\gamma L + \sqrt{\frac{4X(\gamma)}{b}}\,\sqrt{Y}}\, N_0(\gamma) \end{aligned}$$

$\Rightarrow$ dispersion relation at large Y and L:

$$v(\gamma) = \frac{1}{\gamma}\sqrt{\frac{4X(\gamma)}{b}}$$

The effective time for the wave is $\sqrt{Y}$ and not Y.

Critical parameters

$$\begin{aligned} v(\gamma) &= \frac{1}{\gamma}\sqrt{\frac{4X(\gamma)}{b}} \\ X(\gamma) &= \int_{\hat{\gamma}}^{\gamma} \mathrm{d}\gamma' \, \chi_{LL}(\gamma') \end{aligned}$$

$v_c = \min v(\gamma) = v(\gamma_c)$ depend on $\hat{\gamma}$. Let us choose $\hat{\gamma}$ such that

$$\frac{\mathrm{d}v_c(\hat{\gamma})}{\mathrm{d}\hat{\gamma}} = 0$$

$\Rightarrow$ Critical parameters:

$$\begin{aligned} \chi_{LL}(\gamma_c) &= \gamma_c \, \chi'_{LL}(\gamma_c) \qquad \gamma_c \simeq 0.6275 \\ v_c &= \sqrt{\frac{2\chi_{LL}(\gamma_c)}{b\gamma_c}} \end{aligned}$$

Resolution

Expansion of the kernel around $\gamma \sim \gamma_c$

$$\frac{bL}{2\sqrt{Y}}\,\partial_{\sqrt{Y}} N = \left[-\frac{bv_c^2}{2}\,\partial_L + \frac{1}{2}\,\chi''_{LL}(\gamma_c)(\partial_L^2 + 2\gamma_c\,\partial_L + \gamma_c^2) + \dots\right] N$$

Then, using the same Ansatz:

$$\begin{aligned} N(L,Y) &= A\,Y^{1/6}\,\mathrm{Ai}\left(\bar{\xi}_1 + \left(\frac{\sqrt{2b\gamma_c\chi_{LL}(\gamma_c)}}{\chi''_{LL}(\gamma_c)}\right)^{1/3}\frac{\xi}{Y^{1/6}}\right)\,e^{-\gamma_c\xi} \\ \xi &\equiv \log\left(\frac{k^2}{Q_s^2(Y)}\right) \\ &= L - \sqrt{\frac{2\chi_{LL}(\gamma_c)Y}{b\gamma_c}} - \frac{3\bar{\xi}_1}{4}\left(\frac{\chi''_{LL}(\gamma_c)}{\sqrt{2b\gamma_c\chi_{LL}(\gamma_c)}}\right)^{\frac{1}{3}} Y^{\frac{1}{6}} + \mathcal{O}(Y^{-\frac{1}{6}}) \end{aligned}$$

BK at NLL

General form:

$$\begin{aligned}\partial_Y N(L,Y) &= \bar{\alpha}\left[\chi_{LL}(-\partial_L) + \bar{\alpha}\chi_{NLL}(-\partial_L)\right] N(L,Y) \\ &\quad -\bar{\alpha}\left[N^2(L,Y) + \bar{\alpha}\left(\text{NLL nonlinear terms}\right)\right] \\ &\quad +\bar{\alpha}^2\left(\text{New terms ?}\right)\end{aligned}$$

Collecting the running coupling terms:

$$\begin{aligned}\partial_Y N(L,Y) &= \bar{\alpha}(L)\left[\chi_{LL}(-\partial_L) + \bar{\alpha}(L)\chi^{(1)}(-\partial_L)\right] N(L,Y) \\ &\quad -\bar{\alpha}(L)\left(\text{Nonlinear terms}\right) \\ &\quad +\bar{\alpha}^2(L)\left(\text{New terms ?}\right)\end{aligned}$$

Effective linear equation

Linear part of the NLL equation:

$$\partial_Y N(L,Y) \quad = \quad \frac{1}{bL}\left[\chi_{LL}(-\partial_L) + \frac{1}{bL}\chi^{(1)}(-\partial_L)\right] N(L,Y)$$

Is it equivalent to an effective equation

$$\partial_Y N(L,Y) \quad = \quad \frac{1}{bL}\kappa(-\partial_L, \partial_Y) N(L,Y)\ ?$$

Ciafaloni, Colferai (1998)

Effective linear equation

In Laplace (or Mellin) space:

$$-\partial_L \to \gamma \quad \text{and} \quad \partial_Y \to \omega$$

$$\begin{aligned} \omega &= \frac{1}{bL}\left[\chi_{LL}(\gamma) + \frac{1}{bL}\chi^{(1)}(\gamma)\right] \\ \omega &= \frac{1}{bL}\,\kappa(\gamma,\omega) \end{aligned}$$

They are equivalent if

$$\kappa(\gamma,\omega) = \chi_{LL}(\gamma) + \frac{\omega\;\chi^{(1)}(\gamma)}{\kappa(\gamma,\omega)}\;.$$

Omega expansion

Effective kernel:

$$\begin{aligned} \kappa(\gamma,\omega) &= \chi_{LL}(\gamma)\,\frac{1}{2}\left[1+\sqrt{1+\frac{4\omega\;\chi^{(1)}(\gamma)}{\chi_{LL}^2(\gamma)}}\right] \\ &\simeq \chi_{LL}(\gamma)\left[1+\frac{\omega\;\chi^{(1)}(\gamma)}{\chi_{LL}^2(\gamma)}-\left(\frac{\omega\;\chi^{(1)}(\gamma)}{\chi_{LL}^2(\gamma)}\right)^2+\ldots\right] \end{aligned}$$

The effective equation can be trusted if

$$\frac{\omega\;\chi^{(1)}(\gamma)}{\chi_{LL}^2(\gamma)} \ll 1$$

Omega expansion

The LL and NLL kernel eigenfunctions have singularities in $\gamma \to 0$ (or $1-\gamma \to 0$)

$$\begin{array}{rcll} \chi_{LL}(\gamma) & \propto & \gamma^{-1} & (\text{or}(1-\gamma)^{-1}) \\ \chi^{(1)}(\gamma) & \propto & \gamma^{-3} & (\text{or}(1-\gamma)^{-3}) \end{array}$$

$$\frac{\omega\, \chi^{(1)}(\gamma)}{\chi^2_{LL}(\gamma)} \propto \frac{\omega}{\gamma} \quad \left(\text{or}\, \frac{\omega}{1-\gamma}\right)$$

$\Rightarrow$ The effective equation is valid if $\omega \ll \gamma, 1-\gamma$

Approximate linear solution

$$N(L,Y) \;=\; \int \frac{\mathrm{d}\gamma}{2\pi i} \int \frac{\mathrm{d}\omega}{2\pi i}\, e^{-\gamma L+\omega Y+\frac{1}{b\omega}\,X(\gamma,\omega)}\, N_0(\gamma,\omega)$$

with $$X(\gamma,\omega) = \int_{\hat\gamma}^{\gamma} \mathrm{d}\gamma'\;\kappa(\gamma',\omega)$$

is an approximate solution of the linear equation at large L, because the large L saddle point equation is

$$L = \frac{1}{b\omega}\,\kappa(\gamma,\omega)$$

which is equivalent to

$$\omega = \frac{1}{bL}\left[\chi_{LL}(\gamma) + \frac{1}{bL}\chi^{(1)}(\gamma)\right] \quad \text{if } \omega \ll \gamma, 1-\gamma\,.$$

Large Y saddle point

$$N(L,Y) = \int \frac{\mathrm{d}\gamma}{2\pi i} \int \frac{\mathrm{d}\omega}{2\pi i}\, e^{-\gamma L + \omega Y + \frac{1}{b\omega}\ X(\gamma,\omega)}\ N_0(\gamma,\omega)$$

At large Y, the saddle point approximation in ω gives

$$\begin{aligned} Y b \omega_s^2 &= X(\gamma,\omega_s) - \omega_s\, \dot{X}(\gamma,\omega_s) \\ &= \int_{\hat{\gamma}}^{\gamma} \mathrm{d}\gamma' \chi_{LL}(\gamma') \left[1 + \left(\frac{\omega_s\, \chi^{(1)}(\gamma)}{\chi_{LL}^2(\gamma)} \right)^2 \right. \\ &\qquad \left. -4 \left(\frac{\omega_s\, \chi^{(1)}(\gamma)}{\chi_{LL}^2(\gamma)} \right)^3 + \ldots \right] \end{aligned}$$

Critical parameters

If $\gamma \to 0$:

$$Y b\omega_s^2 \quad = \quad \log\gamma + \text{constant} - \frac{1}{2}\left(\frac{C\omega_s}{\gamma}\right)^2 + \frac{4}{3}\left(\frac{C\omega_s}{\gamma}\right)^3 + \ldots$$

For large Y, $\omega_s \propto Y^{-1/2}$, and the higher orders are suppressed if $\omega_s \ll \gamma\,,\, 1-\gamma$.

$\Rightarrow$ same γ_c and v_c as in the simplest equation with running coupling. And the convergence can start when

$$Y > \gamma_c^{-2}\,,\;(1-\gamma_c)^{-2}$$

Expansion of the effective equation

Expanding the effective kernel around $\gamma \sim \gamma_c$ and $\omega \sim 0$:

$$\frac{bL}{2\sqrt{Y}}\,\partial_{\sqrt{Y}} N = \Bigg(-\frac{bv_c^2}{2}\,\partial_L + \frac{1}{2}\,\chi''_{LL}(\gamma_c)(\partial_L^2 + 2\gamma_c\,\partial_L + \gamma_c^2) + \dots$$

$$+\frac{1}{2\sqrt{Y}}\,\frac{\chi^{(1)}(\gamma_c)}{\chi_{LL}(\gamma_c)}\,\partial_{\sqrt{Y}} - \frac{1}{2\sqrt{Y}}\left(\partial_\gamma \frac{\chi^{(1)}(\gamma)}{\chi_{LL}(\gamma)}\right)_{\gamma=\gamma_c}(\gamma_c + \partial_L)\,\partial_{\sqrt{Y}}$$

$$+ \dots \Bigg) N$$

The new terms, of order $Y^{-1/2}$ doesn't contribute to the universal subasymptotic behavior.

Result at NLL

At large enough rapidity ($Y^{-1/2} \ll 1$), the solution of the full NLL BK equation in the geometric scaling region converge to the solution of the LL BK equation with running coupling. Then, they converge to their asymptotic solution.

$$
\begin{aligned}
N(L,Y) &= A\, Y^{1/6}\, \mathrm{Ai}\left(\bar{\xi}_1 + \left(\frac{\sqrt{2b\gamma_c\chi_{LL}(\gamma_c)}}{\chi''_{LL}(\gamma_c)}\right)^{1/3} \frac{\xi}{Y^{1/6}}\right) e^{-\gamma_c\xi} \\
\xi &\equiv \log\left(\frac{k^2}{Q_s^2(Y)}\right) \\
&= L - \sqrt{\frac{2\chi_{LL}(\gamma_c)Y}{b\gamma_c}} - \frac{3\bar{\xi}_1}{4}\left(\frac{\chi''_{LL}(\gamma_c)}{\sqrt{2b\gamma_c\chi_{LL}(\gamma_c)}}\right)^{\frac{1}{3}} Y^{\frac{1}{6}}
\end{aligned}
$$